Preprint typeset in JHEP style. - HYPER VERSION

MIT-CTP-3026

hep-th/0010023

Discrete Torsion, Non-Abelian Orbifolds and the Schur Multiplier

Bo Feng, Amihay Hanany, Yang-Hui He and Nikolaos Prezas *

Center for Theoretical Physics,
Massachusetts Institute of Technology,
Cambridge, MA 02139, USA.
fengb, hanany, yhe, prezas@ctp.mit.edu

Abstract: Armed with the explicit computation of Schur Multipliers, we offer a classification of $SU(n)$ orbifolds for $n = 2, 3, 4$ which permit the turning on of discrete torsion. This is in response to the host of activity lately in vogue on the application of discrete torsion to D-brane orbifold theories. As a by-product, we find a hitherto unknown class of $\mathcal{N} = 1$ orbifolds with non-cyclic discrete torsion group. Furthermore, we supplement the *status quo ante* by investigating a first example of a non-Abelian orbifold admitting discrete torsion, namely the ordinary dihedral group as a subgroup of $SU(3)$. A comparison of the quiver theory thereof with that of its covering group, the binary dihedral group, without discrete torsion, is also performed.

Keywords: Discrete Torsion, non-Abelian Orbifolds, Schur Multiplier, Discrete Subgroups of SU(2), SU(3) and SU(4).

*Research supported in part by the Reed Fund Award, the CTP and the LNS of MIT and the U.S. Department of Energy under cooperative research agreement # DE-FC02-94ER40818. A. H. is also supported by an A. P. Sloan Foundation Fellowship, and a DOE OJI award.

Contents

1. Introduction

The study of string theory in non-trivial NS-NS B-field backgrounds has of late become one of the most pursued directions of research. Ever since the landmark papers [1], where it was shown that in the presence of such non-trivial B-fields along the world-volume directions of the D-brane, the gauge theory living thereupon assumes a non-commutative guise in the large-B-limit, most works were done in this direction of space-time non-commutativity. However, there is an alternative approach in the investigation of the effects of the B-field, namely **discrete torsion**, which is of great interest in this respect. On the other hand, as discrete torsion presents itself to be a natural generalisation to the study of orbifold projections of D-brane probes at space-time singularities, a topic under much research over the past few years, it is also mathematically and physically worthy of pursuit under this light.

A brief review of the development of the matter from a historical perspective shall serve to guide the reader. Discrete torsion first appeared in [2] in the study of the closed string partition function $Z(q,\bar{q})$ on the orbifold G. And shortly thereafter, it effects on the geometry of space-time were pointed out [3]. In particular, [2] noticed that $Z(q,\bar{q})$ could contain therein, phases $\epsilon(g,h) \in U(1)$ for $g,h \in G$, coming from the twisted sectors of the theory, as long as

$$\begin{aligned} &\epsilon(g_1 g_2, g_3) = \epsilon(g_1, g_3)\epsilon(g_2, g_3) \\ &\epsilon(g,h) = 1/\epsilon(h,g) \\ &\epsilon(g,g) = 1, \end{aligned} \tag{1.1}$$

so as to ensure modular invariance.

Reviving interests along this line, Douglas and Fiol [4, 5] extended discrete torsion to the open string sector by showing that the usual procedure of projection by orbifolds on D-brane probes [6, 7], applied to **projective representations** instead of the ordinary *linear representations* of the orbifold group G, gives exactly the gauge theory with discrete torsion turned on. In other words, for the invariant matter fields which survive the orbifold, Φ such that $\gamma^{-1}(g)\Phi\gamma(g) = r(g)\Phi, \quad \forall\ g \in G$, we now need the representation

$$\begin{aligned} &\gamma(g)\gamma(h) = \alpha(g,h)\gamma(gh), \quad g,h \in G \ \text{ with} \\ &\alpha(x,y)\alpha(xy,z) = \alpha(x,yz)\alpha(y,z), \qquad \alpha(x,\mathbb{I}_G) = 1 = \alpha(\mathbb{I}_G,x) \quad \forall x,y,z \in G, \end{aligned} \tag{1.2}$$

where $\alpha(g,h)$ is known as a cocycle. These cocycles constitute, up to the equivalence

$$\alpha(g,h) \sim \frac{c(g)c(h)}{c(gh)}\alpha(g,h), \tag{1.3}$$

the so-called second cohomology group $H^2(G,U(1))$ of G, where c is any function (not necessarily a homomorphism) mapping G to $U(1)$; this is what we usually mean by *discrete torsion*

being classified by $H^2(G, U(1))$. We shall formalise all these definitions in the subsequent sections.

In fact, one can show [2], that the choice

$$\epsilon(g,h) = \frac{\alpha(g,h)}{\alpha(h,g)},$$

for α obeying (1.2) actually satisfies (1.1), whereby linking the concepts of discrete torsion in the closed and open string sectors. We point this out as one could be easily confused as to the precise parametre called discrete torsion and which is actually classified by the second group cohomology.

Along the line of [4, 5], a series of papers by Berenstein, Leigh and Jejjala [8, 9] developed the technique to study the *non-commutative* moduli space of the $\mathcal{N} = 1$ gauge theory living on $\mathbb{C}^3/\mathbb{Z}_m \times \mathbb{Z}_n$ parametrised as an algebraic variety. A host of activities followed in the generalisation of this abelian orbifold, notably to $\mathbb{C}^4/\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ by [10], to the inclusion of orientifolds by [11], and to the orbifolded conifold by [12].

Along the mathematical thread, Sharpe has presented a prolific series of works to relate discrete torsion with connection on gerbes [14], which could allow generalisations of the concept to beyond the 2-form B-field. Moreover, in relation to twisted K-theory and attempts to unify space-time cohomology with group cohomology in the vein of the McKay Correspondence (see e.g. [15]), works by Gomis [16] and Aspinwall-Plesser [17, 18] have given some guiding light.

Before we end this review of the current studies, we would like to mention the work by Gaberdiel [13]. He pointed out that there exists a different choice, such that the original intimate relationship between discrete torsion in the closed string sector and the non-trivial cocycle in the open sector can be loosened. It would be interesting to investigate further in this spirit.

We see however, that during these last three years of renewed activity, the focus has mainly been on Abelian orbifolds. It is one of the main intentions of this paper to initiate the study of non-Abelian orbifolds with discrete torsion, which, to the best of our knowledge, have not been discussed so far in the literature[2]. We shall classify the general orbifold theories with $\mathcal{N} = 0, 1, 2$ supersymmetry which *could allow discrete torsion* by exhaustively computing the second cohomology of the discrete subgroups of $SU(n)$ for $n = 4, 3, 2$.

Thus rests the current state of affairs. Our main objectives are two-fold: to both supplement the past, by presenting and studying a first example of a non-Abelian orbifold which affords discrete torsion, and to presage the future, by classifying the orbifold theories which could allow discrete torsion being turned on.

[2]In the context of conformal field theory on orbifolds, there has been a recent work addressing some non-Abelian cases [31].

Nomenclature

Throughout this paper, unless otherwise specified, we shall adhere to the following conventions for notation:

ω_n	n-th root of unity;
G	finite group of order $\|G\|$;
$\mathbb{F}$	(algebraically closed) number field;
$\mathbb{F}^*$	multiplicative subgroup of $\mathbb{F}$;
$\langle x_i \| y_j \rangle$	the group generated by elements $\{x_i\}$ with relations y_j;
$< G_1, G_2, \ldots, G_n >$	group generated by the generators of groups $G_1, G_2, \ldots, G_n$;
$\gcd(m,n)$	the greatest common divisor of m and n;
$D_{2n}, E_{6,7,8}$	ordinary dihedral, tetrahedral, octahedral and icosahedral groups;
$\widehat{D_{2n}}, \widehat{E_{6,7,8}}$	the binary counterparts of the above;
A_n and S_n	alternating and symmetric groups on n elements;
$H \triangleleft G$	H is a normal subgroup of G;
$A \rtimes B$	semi-direct product of A and B;
$Z(G)$	centre of G;
$N_G(H)$	the normaliser of $H \subset G$;
$G' := [G,G]$	the derived (commutator) group of G;
$\exp(G)$	exponent of group G.

2. Some Mathematical Preliminaries

2.1 Projective Representations of Groups

We begin by first formalising (1.2), the group representation of our interest:

DEFINITION 2.1 *A* **projective** *representation of G over a field $\mathbb{F}$ (throughout we let $\mathbb{F}$ be an algebraically closed field with characteristic $p \geq 0$) is a mapping $\rho : G \to GL(V)$ such that*

$$(A) \quad \rho(x)\rho(y) = \alpha(x,y)\rho(xy) \ \ \forall \ \ x,y \in G; \qquad (B) \quad \rho(\mathbb{I}_G) = \mathbb{I}_V.$$

Here $\alpha : G \times G \to \mathbb{F}^*$ is a mapping whose meaning we shall clarify later. Of course we see that if $\alpha = 1$ trivially, then we have our familiar ordinary representation of G to which we shall refer as *linear*. Indeed, the mapping ρ into $GL(V)$ defined above is naturally equivalent to a homomorphism into the projective linear group $PGL(V) \cong GL(V)/\mathbb{F}^*\mathbb{I}_V$, and hence the name "projective." In particular we shall be concerned with projective *matrix* representations of G where we take $GL(V)$ to be $GL(n,\mathbb{F})$.

The function α can not be arbitrary and two immediate restrictions can be placed thereupon purely from the structure of the group:

$$\begin{array}{lll} (a) \text{ Group Associativity} & \Rightarrow \alpha(x,y)\alpha(xy,z) = \alpha(x,yz)\alpha(y,z), & \forall x,y,z \in G \\ (b) \text{ Group Identity} & \Rightarrow \alpha(x,\mathbb{I}_G) = 1 = \alpha(\mathbb{I}_G,x), & \forall x \in G. \end{array} \tag{2.1}$$

These conditions on α naturally leads to another discipline of mathematics.

2.2 Group Cohomology and the Schur Multiplier

The study of such functions on a group satisfying (2.1) is precisely the subject of the theory of **Group Cohomology**. In general we let α to take values in A, an abelian coefficient group ($\mathbb{F}^*$ is certainly a simple example of such an A) and call them **cocycles**. The set of all cocycles we shall name $Z^2(G,A)$. Indeed it is straight-forward to see that $Z^2(G,A)$ is an abelian group. We subsequently define a set of functions satisfying

$$B^2(G,A) := \{(\delta g)(x,y) := g(x)g(y)g(xy)^{-1}\} \quad \text{for any } g : G \to A \text{ such that } g(\mathbb{I}_G) = 1, \tag{2.2}$$

and call them *coboundaries*. It is then obvious that $B^2(G,A)$ is a (normal) subgroup of $Z^2(G,A)$ and in fact constitutes an equivalence relation on the latter in the manner of (1.3). Thus it becomes a routine exercise in cohomology to define

$$H^2(G,A) := Z^2(G,A)/B^2(G,A),$$

the *second cohomology* group of G.

Summarising what we have so far, we see that the projective representations of G are classified by its second cohomology $H^2(G,\mathbb{F}^*)$. To facilitate the computation thereof, we shall come to an important concept:

DEFINITION 2.2 *The* **Schur Multiplier** $M(G)$ *of the group* G *is the second cohomology group with respect to the trivial action of* G *on* $\mathbb{C}^*$*:*

$$M(G) := H^2(G,\mathbb{C}^*).$$

Since we shall be mostly concerned with the field $\mathbb{F} = \mathbb{C}$, the Schur multiplier is exactly what we need. However, the properties thereof are more general. In fact, for any algebraically closed field $\mathbb{F}$ of zero characteristic, $M(G) \cong H^2(G,\mathbb{F}^*)$. In our case of $\mathbb{F} = \mathbb{C}$, it can be shown that [11],

$$H^2(G,\mathbb{C}^*) \cong H^2(G,U(1)).$$

This terminology is the more frequently encountered one in the physics literature.

One task is thus self-evident: the calculation of the Schur Multiplier of a given group G shall indicate possibilities of projective representations of the said group, or in a physical

language, the possibilities of turning on discrete torsion in string theory on the orbifold group G. In particular, if $M(G) \cong \mathbb{I}$, then the second cohomology of G is trivial and no non-trivial discrete torsion is allowed. We summarise this

$$\text{KEY POINT:} \quad \text{Calculate } M(G) \Rightarrow \text{Information on Discrete Torsion.}$$

2.3 The Covering Group

The study of the actual projective representation of G is very involved and what is usually done in fact is to "lift to an ordinary representation." What this means is that for a central extension[3] A of G to G^*, we say a projective representation ρ of G **lifts** to a linear representation ρ^* of G^* if (i) $\rho^*(a \in A)$ is proportional to $\mathbb{I}$ and (ii) there is a section[4] $\mu : G \to G^*$ such that $\rho(g) = \rho^*(\mu(g)), \quad \forall g \in G$. Likewise it *lifts projectively* if $\rho(g) = t(g)\rho^*(\mu(g))$ for a map $t : G \to \mathbb{F}^*$. Now we are ready to give the following:

DEFINITION 2.3 *We call G^* a* **covering group**[5] *of G over $\mathbb{F}$ if the following are satisfied:*
(i) $\exists$ a central extension $1 \to A \to G^* \to G \to 1$ such that any projective representation of G lifts projectively to an ordinary representation of G^*;
(ii) $|A| = |H^2(G, \mathbb{F}^*)|$.

The following theorem, initially due to Schur, characterises covering groups.

THEOREM 2.1 ([20] p143) *$G^\star$ is a covering group of G over $\mathbb{F}$ if and only if the following conditions hold:*
(i) $G^\star$ has a finite subgroup A with $A \subseteq Z(G^\star) \cap [G^\star, G^\star]$;
(ii) $G \cong G^\star/A$;
(iii) $|A| = |H^2(G, F^\star)|$
where $[G^\star, G^\star]$ is the derived group[6] $G^{*'}$ of G^*.

Thus concludes our prelude on the mathematical rudiments, the utility of the above results shall present themselves in the ensuing.

3. Schur Multipliers and String Theory Orbifolds

The game is thus afoot. Orbifolds of the form $\mathbb{C}^k/\{G \in SU(k)\}$ have been widely studied in the context of gauge theories living on D-branes probing the singularities. We need only to compute $M(G)$ for the discrete finite groups of $SU(n)$ for $n = 2, 3, 4$ to know the discrete torsion afforded by the said orbifold theories.

[3] i.e., A in the centre $Z(G^*)$ and $G^*/A \cong G$ according to the exact sequence $1 \to A \to G^* \to G \to 1$.
[4] i.e., for the projection $f : G^* \to G$, $\mu \circ f = \mathbb{I}_G$.
[5] Sometimes is also known as **representation group**.
[6] For a group G, $G' := [G, G]$ is the group generated by elements of the form $xyx^{-1}y^{-1}$ for $x, y \in G$.

3.1 The Schur Multiplier of the Discrete Subgroups of $SU(2)$

Let us first remind the reader of the well-known ADE classification of the discrete finite subgroups of $SU(2)$. Here are the presentations of these groups:

G	Name	Order	Presentation
$\widehat{A_n}$	Cyclic, $\cong \mathbb{Z}_{n+1}$	n	$\langle a \vert a^n = \mathbb{I} \rangle$
$\widehat{D_{2n}}$	Binary Dihedral	$4n$	$\langle a, b \vert b^2 = a^n, abab^{-1} = \mathbb{I} \rangle$
$\widehat{E_6}$	Binary Tetrahedral	24	$\langle a, b \vert a^3 = b^3 = (ab)^3 \rangle$
$\widehat{E_7}$	Binary Octahedral	48	$\langle a, b \vert a^4 = b^3 = (ab)^2 \rangle$
$\widehat{E_8}$	Binary Icosahedral	120	$\langle a, b \vert a^5 = b^3 = (ab)^2 \rangle$

(3.1)

We here present a powerful result due to Schur (1907) (q.v. Cor. 2.5, Chap. 11 of [21]) which aids us to explicitly compute large classes of Schur multipliers for finite groups:

THEOREM 3.2 ([20] p383) *Let G be generated by n elements with (minimally) r defining relations and let the Schur multiplier $M(G)$ have a minimum of s generators, then*

$$r \geq n + s.$$

In particular, $r = n$ implies that $M(G)$ is trivial and $r = n + 1$, that $M(G)$ is cyclic.

Theorem 3.2 could be immediately applied to $G \in SU(2)$.

Let us proceed with the computation case-wise. The $\widehat{A_n}$ series has 1 generator with 1 relation, thus $r = n = 1$ and $M(\widehat{A_n})$ is trivial. Now for the $\widehat{D_{2n}}$ series, we note briefly that the usual presentation is $\widehat{D_{2n}} := \langle a, b \vert a^{2n} = \mathbb{I}, b^2 = a^n, bab^{-1} = a^{-1} \rangle$ as in [23]; however, we can see easily that the last two relations imply the first, or explicitly: $a^{-n} := (bab^{-1})^n = ba^n b^{-1} = a^n$, (q.v. [21] Example 3.1, Chap. 11), whence making $r = n = 2$, i.e., 2 generators and 2 relations, and further making $M(\widehat{D_{2n}})$ trivial. Thus too are the cases of the 3 exceptional groups, each having 2 generators with 2 relations. In summary then we have the following corollary of Theorem 3.2, the well-known [17] result that

COROLLARY 3.1 *All discrete finite subgroups of $SU(2)$ have second cohomology $H^2(G, \mathbb{C}^*) = \mathbb{I}$, and hence afford no non-trivial discrete torsion.*

It is intriguing that the above result can actually be hinted from physical considerations without recourse to heavy mathematical machinery. The orbifold theory for $G \subset SU(2)$ preserves an $\mathcal{N} = 2$ supersymmetry on the world-volume of the D3-Brane probe. Inclusion of discrete torsion would deform the coefficients of the superpotential. However, $\mathcal{N} = 2$ supersymmetry is highly restrictive and in general does not permit the existence of such deformations. This is in perfect harmony with the triviality of the Schur Multiplier of $G \subset SU(2)$ as presented in the above Corollary.

To address more complicated groups we need a methodology to compute the Schur Multiplier, and we have many to our aid, for after all the computation of $M(G)$ is a vast subject entirely by itself. We quote one such method below, a result originally due to Schur:

THEOREM 3.3 ([22] p54) *Let $G = F/R$ be the defining finite presentation of G with F the free group of rank n and R is (the normal closure of) the set of relations. Suppose $R/[F,R]$ has the presentation $\langle x_1, \ldots, x_m; y_1, \ldots, y_n \rangle$ with all x_i of finite order, then*

$$M(G) \cong \langle x_1, \ldots, x_n \rangle.$$

Two more theorems of great usage are the following:

THEOREM 3.4 ([22] p17) *Let the exponent*[7] *of $M(G)$ be $\exp(M(G))$, then*

$$\exp(M(G))^2 \text{ divides } |G|.$$

And for direct products, another fact due to Schur,

THEOREM 3.5 ([22] p37)

$$M(G_1 \times G_2) \cong M(G_1) \times M(G_2) \times (G_1 \otimes G_2),$$

where $G_1 \otimes G_2$ is defined to be $\mathrm{Hom}_{\mathbb{Z}}(G_1/G_1', G_2/G_2')$.

With the above and a myriad of useful results (such as the Schur Multiplier for semi-direct products), and especially with the aid of the Computer Algebra package GAP [24] using the algorithm developed for the p-Sylow subgroups of Schur Multiplier [25], we have engaged in the formidable task of giving the explicit Schur Multiplier of the list of groups of our interest.

Most of the details of the computation we shall leave to the appendix, to give the reader a flavour of the calculation but not distracting him or her from the main course of our writing. Without much further ado then, we now proceed with the list of Schur Multipliers for the discrete subgroups of $SU(n)$ for $n = 3, 4$, i.e., the $\mathcal{N} = 1, 0$ orbifold theories.

3.2 The Schur Multiplier of the Discrete Subgroups of $SU(3)$

The classification of the discrete finite groups of $SU(3)$ is well-known (see e.g. [26, 27, 28] for a discussion thereof in the context of string theory). It was pointed out in [23] that the usual classification of these groups does not include the so-called *intransitive* groups (see [29] for definitions), which are perhaps of less mathematical interest. Of course from a physical stand-point, they all give well-defined orbifolds. More specifically [23], all the

[7] i.e., the lowest common multiple of the orders of the elements.

ordinary polyhedral subgroups of $SO(3)$, namely the ordinary dihedral group D_{2n} and the ordinary $E_6 \cong A_4 \cong \Delta(3 \times 2^2), E_7 \cong S_4 \cong \Delta(6 \times 2^2), E_8 \cong \Sigma_{60}$, due to the embedding $SO(3) \hookrightarrow SU(3)$, are obviously (intransitive) subgroups thereof and thus we shall include these as well in what follows. We discuss some aspects of the intransitives in Appendix B and are grateful to D. Berenstein for pointing out some subtleties involved [33]. We insert one more cautionary note. The $\Delta(6n^2)$ series does not actually include the cases for n odd [28]; therefore n shall be restricted to be even.

Here then are the Schur Multipliers of the $SU(3)$ discrete subgroups.

	G	Order	Schur Multiplier $M(G)$
Intransitives	$\mathbb{Z}_n \times \mathbb{Z}_m$	$n \times m$	$\mathbb{Z}_{\gcd(n,m)}$
	$< \mathbb{Z}_n, \widehat{D_{2m}} >$	$\begin{cases} n \times 4m & n \text{ odd} \\ \frac{n}{2} \times 4m & n \text{ even} \end{cases}$	$\begin{cases} \mathbb{I} & n \bmod 4 \neq 1 \\ \mathbb{Z}_2 & n \bmod 4 = 0, m \text{ odd} \\ \mathbb{Z}_2 \times \mathbb{Z}_2 & n \bmod 4 = 0, m \text{ even} \end{cases}$
	$< \mathbb{Z}_n, \widehat{E_6} >$	$\begin{cases} n \times 24 & n \text{ odd} \\ \frac{n}{2} \times 24 & n \text{ even} \end{cases}$	$\mathbb{Z}_{\gcd(n,3)}$
	$< \mathbb{Z}_n, \widehat{E_7} >$	$\begin{cases} n \times 48 & n \text{ odd} \\ \frac{n}{2} \times 48 & n \text{ even} \end{cases}$	$\begin{cases} \mathbb{I} & n \bmod 4 \neq 0 \\ \mathbb{Z}_2 & n \bmod 4 = 0 \end{cases}$
	$< \mathbb{Z}_n, \widehat{E_8} >$	$\begin{cases} n \times 120 & n \text{ odd} \\ \frac{n}{2} \times 120 & n \text{ even} \end{cases}$	$\mathbb{I}$
	Ordinary Dihedral D_{2n}	$2n$	$\mathbb{Z}_{\gcd(n,2)}$
	$< \mathbb{Z}_n, D_{2m} >$	$\begin{cases} n \times 2m & m \text{ odd} \\ n \times 2m & m \text{ even}, n \text{ odd} \\ \frac{n}{2} \times 2m & m \text{ even}, n \text{ even} \end{cases}$	$\begin{cases} \mathbb{Z}_{\gcd(n,2)} & m \text{ odd} \\ \mathbb{Z}_2 & m \text{ even}, n \bmod 4 = 1,2,3 \\ \mathbb{Z}_2 & m \bmod 4 \neq 0, n \bmod 4 = 0 \\ \mathbb{Z}_2 \times \mathbb{Z}_2 & m \bmod 4 = 0, n \bmod 4 = 0 \end{cases}$
Transitives	Δ_{3n^2}	$3n^2$	$\begin{cases} \mathbb{Z}_n \times \mathbb{Z}_3, & \gcd(n,3) \neq 1 \\ \mathbb{Z}_n, & \gcd(n,3) = 1 \end{cases}$
	Δ_{6n^2} (n even)	$6n^2$	$\mathbb{Z}_2$
	$\Sigma_{60} \cong A_5$	60	$\mathbb{Z}_2$
	Σ_{168}	168	$\mathbb{Z}_2$
	Σ_{108}	36×3	$\mathbb{I}$
	Σ_{216}	72×3	$\mathbb{I}$
	Σ_{648}	216×3	$\mathbb{I}$
	Σ_{1080}	360×3	$\mathbb{Z}_2$

(3.2)

Some immediate comments are at hand. The question of whether any discrete subgroup of $SU(3)$ admits non-cyclic discrete torsion was posed in [17]. From our results in table (3.2), we have shown by explicit construction that the answer is in the affirmative: not only the various intransitives give rise to product cyclic Schur Multipliers, so too does the transitive $\Delta(3n^2)$ series for n a multiple of 3.

In Appendix A we shall present the calculation for $M(\Delta_{3n^2})$ and $M(\Delta_{6n^2})$ for illustrative purposes. Furthermore, as an example of non-Abelian orbifolds with discrete torsion, we shall investigate the series of the ordinary dihedral group in detail with applications to physics in mind. For now, for the reader's edification or amusement, let us continue with the $SU(4)$ subgroups.

3.3 The Schur Multiplier of the Discrete Subgroups of $SU(4)$

The discrete finite subgroups of $SL(4,\mathbb{C})$, which give rise to non-supersymmetric orbifold theories, are presented in modern notation in [29]. Using the notation therein, and recalling that the group names in $SU(4) \subset SL(4,\mathbb{C})$ were accompanied with a star (*cit. ibid.*), let us tabulate the Schur Multiplier of the exceptional cases of these particulars (cases XXIX$*$ and XXX$*$ were computed by Prof. H. Pahlings to whom we are grateful):

G	Order	Schur Multiplier $M(G)$
I$*$	60×4	$\mathbb{I}$
II$* \cong \Sigma_{60}$	60	$\mathbb{Z}_2$
III$*$	360×4	$\mathbb{Z}_3$
IV$*$	$\frac{1}{2}7! \times 2$	$\mathbb{Z}_3$
VI$*$	$2^6 3^4 5 \times 2$	$\mathbb{I}$
VII$*$	120×4	$\mathbb{Z}_2$
VIII$*$	120×4	$\mathbb{Z}_2$
IX$*$	720×4	$\mathbb{Z}_2$
X$*$	144×2	$\mathbb{Z}_2 \times \mathbb{Z}_3$
XI$*$	288×2	$\mathbb{Z}_2 \times \mathbb{Z}_3$
XII$*$	288×2	$\mathbb{Z}_2$
XIII$*$	720×2	$\mathbb{Z}_2$
XIV$*$	576×2	$\mathbb{Z}_2 \times \mathbb{Z}_2$
XV$*$	1440×2	$\mathbb{Z}_2$
XVI$*$	3600×2	$\mathbb{Z}_2$
XVII$*$	576×4	$\mathbb{Z}_2$
XVIII$*$	576×4	$\mathbb{Z}_2 \times \mathbb{Z}_3$
XIX$*$	288×4	$\mathbb{I}$
XX$*$	7200×4	$\mathbb{I}$
XXI$*$	1152×4	$\mathbb{Z}_2 \times \mathbb{Z}_2$
XXII$*$	$5 \times 16 \times 4$	$\mathbb{Z}_2$
XXIII$*$	$10 \times 16 \times 4$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
XXIV$*$	$20 \times 16 \times 4$	$\mathbb{Z}_2$
XXV$*$	$60 \times 16 \times 4$	$\mathbb{Z}_2$
XXVI$*$	$60 \times 16 \times 4$	$\mathbb{Z}_2 \times \mathbb{Z}_4$
XXVII$*$	$120 \times 16 \times 4$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
XXVIII$*$	$120 \times 16 \times 4$	$\mathbb{Z}_2$
XXIX$*$	$360 \times 16 \times 4$	$\mathbb{Z}_2 \times \mathbb{Z}_3$
XXX$*$	$720 \times 16 \times 4$	$\mathbb{Z}_2$

(3.3)

4. D_{2n} Orbifolds: Discrete Torsion for a non-Abelian Example

As advertised earlier at the end of subsection 3.2, we now investigate in depth the discrete torsion for a non-Abelian orbifold. The ordinary dihedral group $D_{2n} \cong \mathbb{Z}_n \rtimes \mathbb{Z}_2$ of order $2n$, has the presentation

$$D_{2n} = \langle a, b | a^n = 1, b^2 = 1, bab^{-1} = a^{-1} \rangle.$$

As tabulated in (3.2), the Schur Multiplier is $M(D_{2n}) = \mathbb{I}$ for n odd and $\mathbb{Z}_2$ for n even [20]. Therefore the n odd cases are no different from the ordinary linear representations as studied in [23] since they have trivial Schur Multiplier and hence trivial discrete torsion. On the other hand, for the n even case, we will demonstrate the following result:

PROPOSITION 4.1 *The binary dihedral group $\widehat{D_{2n}}$ of the D-series of the discrete subgroups of $SU(2)$ (otherwise called the generalised quaternion group) is the covering group of D_{2n} when n is even.*

Proof: The definition of the binary dihedral group $\widehat{D_{2n}}$, of order $4n$, is

$$\widehat{D_{2n}} = \langle a, b | a^{2n} = 1, b^2 = a^n, bab^{-1} = a^{-1} \rangle,$$

as we saw in subsection 3.1. Let us check against the conditions of Theorem 2.1. It is a famous result that $\widehat{D_{2n}}$ is the double cover of D_{2n} and whence an $\mathbb{Z}_2$ central extension. First we can see that $A = Z(\widehat{D_{2n}}) = \{1, a^n\} \cong \mathbb{Z}_2$ and condition (ii) is satisfied. Second we find that the commutators are $[a^x, a^y] := (a^x)^{-1}(a^y)^{-1}a^x a^y = 1$, $[a^x b, a^y b] = a^{2(x-y)}$ and $[a^x b, a^y] = a^{2y}$. From these we see that the derived group $[\widehat{D_{2n}}, \widehat{D_{2n}}]$ is generated by a^2 and is thus equal to $\mathbb{Z}_n$ (since a is of order $2n$). An important point is that only when n is even does A belong to $Z(\widehat{D_{2n}}) \cap [\widehat{D_{2n}}, \widehat{D_{2n}}]$. This result is consistent with the fact that for odd n, D_{2n} has trivial Schur Multiplier. Finally of course, $|A| = |H^2(G, \mathbb{F}^*)| = 2$. Thus conditions (i) and (iii) are also satisfied. We therefore conclude that for even n, $\widehat{D_{2n}}$ is the covering group of D_{2n}.

4.1 The Irreducible Representations

With the above Proposition, we know by the very definition of the covering group, that the projective representation of D_{2n} should be encoded in the linear representation of $\widehat{D_{2n}}$, which is a standard result that we can recall from [23]. The latter has four 1-dimensional and $n-1$ 2-dimensional irreps. The matrix representations of these 2-dimensionals for the generic elements a^p, ba^p $(p = 0, ..., 2n-1)$ are given below:

$$a^p = \begin{pmatrix} \omega_{2n}^{lp} & 0 \\ 0 & \omega_{2n}^{-lp} \end{pmatrix} \qquad ba^p = \begin{pmatrix} 0 & i^l \omega_{2n}^{-lp} \\ i^l \omega_{2n}^{lp} & 0 \end{pmatrix}, \tag{4.1}$$

with $l = 1, ..., n-1$; these are denoted as χ_2^l. On the other hand, the four 1-dimensionals are

	$n=$ even				$n=$ odd			
	a^{even}	$a(a^{\text{odd}})$	$b(ba^{\text{even}})$	$ba(ba^{\text{odd}})$	a^{even}	$a(a^{\text{odd}})$	$b(ba^{\text{even}})$	$ba(ba^{\text{odd}})$
χ_1^1	1	1	1	1	1	1	1	1
χ_1^2	1	-1	1	-1	1	-1	ω_4	$-\omega_4$
χ_1^3	1	1	-1	-1	1	1	-1	-1
χ_1^4	1	-1	-1	1	1	-1	$-\omega_4$	ω_4

(4.2)

We can subsequently obtain all irreducible projective representations of D_{2n} from the above (henceforth n will be even). Recalling that $\widehat{D_{2n}}/\{1, a^n\} \cong D_{2n}$ from property (ii) of Theorem 2.1, we can choose one element of each of the transversals of $\widehat{D_{2n}}$ with respect to the $\mathbb{Z}_2$ to be mapped to D_{2n}. For convenience we choose $b^x a^y$ with $x = 0, 1$ and $y = 0, 1, ..., n-1$, a total of $4n/2 = 2n$ elements. Thus we are effectively expressing D_{2n} in terms of $\widehat{D_{2n}}$ elements.

For the matrix representation of $a^n \in \widehat{D_{2n}}$, there are two cases. In the first, we have $a^n = 1 \times I_{d\times d}$ where d is the dimension of the representation. This case includes all four 1-dimensional representations and $(n/2-1)$ 2-dimensional representations in (4.1) for $l = 2, 4, ..., n-2$. Because a^n has the same matrix form as $\mathbb{I}$, we see that the elements $b^x a^y$ and $b^x a^{y+n}$ also have the same matrix form. Consequently, when we map them to D_{2n}, they automatically give the irreducible linear representations of D_{2n}.

In the other case, we have $a^n = -1 \times I_{d\times d}$ and this happens when $l = 1, 3, ..., n-1$. It is precisely these cases[8] which give the *irreducible projective representations* of D_{2n}. Now, because a^n has a different matrix form from $\mathbb{I}$, the matrices for $b^x a^y$ and $b^x a^{y+n}$ differ. Therefore, when we map $\widehat{D_{2n}}$ to D_{2n}, there is an ambiguity as to which of the matrix forms, $b^x a^y$ or $b^x a^{y+n}$, to choose as those of D_{2n}.

This ambiguity is exactly a feature of projective representations. Preserving the notations of Theorem 2.1, we let $G^* = \bigcup\limits_{g_i \in G} A g_i$ be the decomposition into transversals of G for the normal subgroup A. Then choosing one element in every transversal, say $A_q g_i$ for some fixed q, we have the ordinary (linear) representation thereof being precisely the projective representation of g_i. Of course different choices of A_q give different but projectively equivalent (projective) representations of G.

By this above method, we can construct all irreducible projective representations of D_{2n} from (4.1). We can verify this by matching dimensions: we end up with $n/2$ 2-dimensional representations inherited from $\widehat{D_{2n}}$ and $2^2 \times (n/2) = 2n$, which of course is the order of D_{2n} as it should.

[8]Sometimes also called **negative representations** in such cases.

4.2 The Quiver Diagram and the Matter Content

The projection for the matter content Φ is well-known (see e.g., [7, 27]):

$$\gamma^{-1}(g)\Phi\gamma(g) = r(g)\Phi, \tag{4.3}$$

for $g \in G$ and r, γ appropriate (projective) representations. The case of D_{2n} without torsion was discussed as a new class of non-chiral $\mathcal{N} = 1$ theories in [23]. We recall that for the group D_{2n} we choose the generators (with action on $\mathbb{C}^3$) as

$$a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega_n & 0 \\ 0 & 0 & \omega_n^{-1} \end{pmatrix} \qquad b = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}. \tag{4.4}$$

Now we can use these explicit forms to work out the matter content (the quiver diagram) and superpotential. For the regular representation, we choose $\gamma(g)$ as block-diagonal in which every 2-dimensional irreducible representation repeats twice with labels $l = 1, 1, 3, 3, .., n-1, n-1$ (as we have shown in the previous section that the even labels correspond to the linear representation of D_{2n}). With this $\gamma(g)$, we calculate the matter content below.

For simplicity, in the actual calculation we would not use (4.3) but rather the standard method given by Lawrence, Nekrasov and Vafa [7], generalised appropriately to the projective case by [17]. We can do so because we are armed with Definition 2.3 and results from the previous subsection, and directly use the linear representation of the covering group: we lift the action of D_{2n} into the action of its covering group $\widehat{D_{2n}}$. It is easy to see that we get the same matter content either by using the projective representations of the former or the linear representations of the latter.

From the point of view of the covering group, the representation $r(g)$ in (4.3) is given by

$$\mathbf{3} \longrightarrow \chi_1^3 + \chi_2^2 \tag{4.5}$$

and the representation $\gamma(g)$ is given by $\gamma \longrightarrow \sum_{l=0}^{n/2-1} 2\chi_2^{2l+1}$. We remind ourselves that the $\mathbf{3}$ must in fact be a *linear* representation of D_{2n} while $\gamma(g)$ is the one that has to be *projective* when we include discrete torsion [4].

For the purpose of tensor decompositions we recall the result for the binary dihedral group [23]:

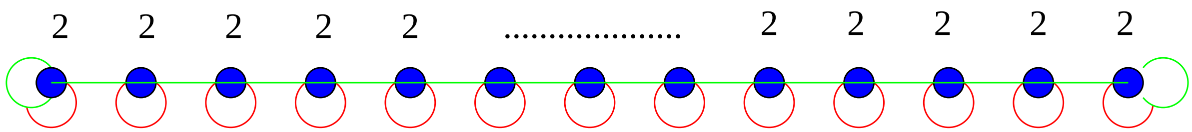

Figure 1: The quiver diagram of the ordinary dihedral group D_{2n} with non-trivial projective representation. In this case of discrete torsion being turned on, we have a product of $n/2$ $U(2)$ gauge groups (nodes). The line connecting two nodes without arrows means that there is one chiral multiplet in each direction. Therefore we have a non-chiral theory.

	$n=$ even	$n=$ odd
$\mathbf{1}\otimes\mathbf{1}'$	$\chi_1^2\chi_1^2=\chi_1^1$ $\chi_1^3\chi_1^3=\chi_1^1$ $\chi_1^4\chi_1^4=\chi_1^1$ $\chi_1^2\chi_1^3=\chi_1^4$ $\chi_1^2\chi_1^4=\chi_1^3$ $\chi_1^3\chi_1^4=\chi_1^2$	$\chi_1^2\chi_1^2=\chi_1^3$ $\chi_1^3\chi_1^3=\chi_1^1$ $\chi_1^4\chi_1^4=\chi_1^3$ $\chi_1^2\chi_1^3=\chi_1^4$ $\chi_1^2\chi_1^4=\chi_1^1$ $\chi_1^3\chi_1^4=\chi_1^2$
$\mathbf{1}\otimes\mathbf{2}$	$\chi_1^h\chi_2^l=\begin{cases}\chi_2^l & h=1,3\\ \chi_2^{n-l} & h=2,4\end{cases}$	
$\mathbf{2}\otimes\mathbf{2}'$	$\chi_2^{l_1}\chi_2^{l_2}=\chi_2^{(l_1+l_2)}+\chi_2^{(l_1-l_2)}$ where	$\chi_2^{(l_1+l_2)}=\begin{cases}\chi_2^{(l_1+l_2)} & \text{if} \quad l_1+l_2<n,\\ \chi_2^{2n-(l_1+l_2)} & \text{if} \quad l_1+l_2>n,\\ \chi_1^2+\chi_1^4 & \text{if} \quad l_1+l_2=n.\end{cases}$ $\chi_2^{(l_1-l_2)}=\begin{cases}\chi_2^{(l_1-l_2)} & \text{if} \quad l_1>l_2,\\ \chi_2^{(l_2-l_1)} & \text{if} \quad l_1<l_2,\\ \chi_1^1+\chi_1^3 & \text{if} \quad l_1=l_2.\end{cases}$

(4.6)

From these relations we immediately obtain the matter content. Firstly, there are $n/2$ $U(2)$ gauge groups ($n/2$ nodes in the quiver). Secondly, because $\chi_1^3\chi_2^l=\chi_2^l$ we have one adjoint scalar for every gauge group. Thirdly, since $\chi_2^2\chi_2^{2l+1}=\chi_2^{2l-1}+\chi_2^{2l+3}$ (where for $l=0$, χ_2^{2l-1} is understood to be χ_2^1 and for $l=n/2-1$, χ_2^{2l+3} is understood to be χ_2^{n-1}), we have two bi-fundamental chiral supermultiplets. We summarise these results in Figure 1.

We want to emphasize that by lifting to the covering group, in general we not only find the matter content (quiver diagram) as we have done above, but also the superpotential as well. The formula is given in (2.7) of [7], which could be applied here without any modification (of course, one can use the matrix form of the group elements to obtain the superpotential directly as done in [4, 5, 6, 8, 9, 10, 11], but (2.7), expressed in terms of the Clebsh-Gordan coefficients, is more convenient).

Knowing the above quiver (cf. Figure 1) of the ordinary dihedral group D_{2n} *with* discrete torsion, we wish to question ourselves as to the relationships between this quiver and that of its covering group, the binary dihedral group $\widehat{D_{2n}}$ *without* discrete torsion (as well as that of D_{2n} without discrete torsion). The usual quiver of $\widehat{D_{2n}}$ is well-known [30, 27]; we give an example for $n=4$ in part (a) of Figure 2. The quiver is obtained by choosing the decomposition of $\mathbf{3}\longrightarrow\chi_1^1+\chi_2^1$ (as opposed to (4.5) because this is the linear representation

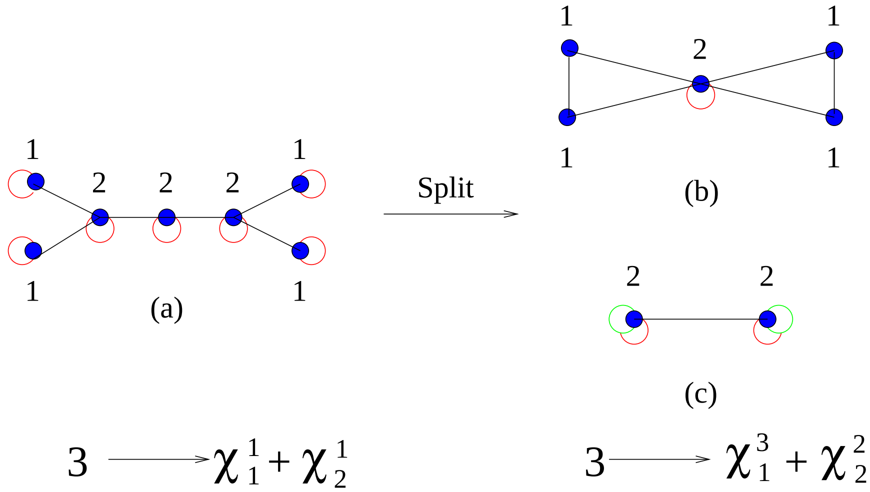

Figure 2: (a) The quiver diagram of the binary dihedral group $\widehat{D}_4$ *without* discrete torsion; (b) the quiver of the ordinary dihedral group D_4 *without* discrete torsion; (c) the quiver of the ordinary dihedral group D_4 *with* discrete torsion.

of $\widehat{D_{2n}}$); also $\gamma(g)$ is in the regular representation of dimension $4n$. A total of $(n-1)+4 = n+3$ nodes results. We recall that when getting the quiver of D_{2n} with discrete torsion in the above, we chose the decomposition of $\mathbf{3} \longrightarrow \chi_1^3 + \chi_2^2$ in (4.5) which provided a linear representation of D_{2n}. Had we made this same choice for $\widehat{D_{2n}}$, our familiar quiver of $\widehat{D_{2n}}$ would have split into two parts: one being precisely the quiver of D_{2n} without discrete torsion as discussed in [23] and the other, that of D_{2n} with discrete torsion as presented in Figure 1. These are given respectively in parts (b) and (c) of Figure 2.

From this discussion, we see that in some sense discrete torsion is connected with different choices of decomposition in the usual orbifold projection. We want to emphasize that the example of D_{2n} is very special because its covering group $\widehat{D}_{2n}$ belongs to $SU(2)$. In general, the covering group does not even belong to $SU(3)$ and the meaning of the usual orbifold projection of the covering group in string theory is vague.

5. Conclusions and Prospects

Let us pause here awhile for reflection. A key purpose of this writing is to initiate the investigation of discrete torsion for the generic D-brane orbifold theories. Inspired by this goal, we have shown that computing the Schur Multiplier $M(G)$ for the finite group G serves as a beacon in our quest.

In particular, using the fact that $M(G)$ is an indicator of when we can turn on a non-trivial NS-NS background in the orbifold geometry and when we cannot: only when $M(G)$, as an Abelian group is not trivially $\mathbb{I}$ can the former be executed. As a guide for future

investigations, we have computed $M(G)$ for the discrete subgroups G in $SU(n)$ with $n = 2, 3, 4$, which amounts to a *classification of which D-brane orbifolds afford non-trivial discrete torsion.*

As an explicit example, in supplementing the present lack of studies of non-Abelian orbifolds with discrete torsion in the current literature, we have pursued in detail the $\mathcal{N} = 1$ gauge theory living on the D3-Brane probe on the orbifold singularity $\mathbb{C}^3/D_{2n}$, corresponding to the ordinary dihedral group of order $2n$ as a subgroup of $SU(3)$. As the group has Schur Multiplier $\mathbb{Z}_2$ for even n, we have turned on the discrete torsion and arrived at an interesting class of non-chiral theories.

The prospects are as manifold as the interests are diverse and much work remains to be done. An immediate task is to examine the gauge theory living on the world-volume of D-brane probes when we turn on the discrete torsion of a given orbifold wherever allowed by our classification. This investigation is currently in progress.

Our results of the Schur Multipliers could also be interesting to the study of K-theory in connexion to string theory. Recent works [16, 17, 19] have noticed an intimate relation between twisted K-theory and discrete torsion. More specifically, the Schur Multiplier of an orbifold group may in fact supply information about the torsion subgroup of the cohomology group of space-time in the light of a generalised McKay Correspondence [17, 15].

It is also tempting to further study the non-commutative moduli space of non-Abelian orbifolds in the spirit of [5, 8, 9] which treated Abelian cases at great length. How the framework developed therein extends to the non-Abelian groups should be interesting. Works on discrete torsion in relation to permutation orbifolds and symmetric products [32] have also been initiated, we hope that our methodologies could be helpful thereto.

Finally, there is another direction of future study. The boundary state formalism was used in [13] where it was suggested that the ties between close and open string sectors maybe softened with regard to discrete torsion. It is thus natural to ask if such ambiguities may exist also for non-Abelian orbifolds.

All these open issues, of concern to the physicist and the mathematician alike, present themselves to the intrigue of the reader.

Acknowledgements

Ad Catharinae Sanctae Alexandriae et Ad Majorem Dei Gloriam...
We would like to acknowledge Professors H. Pahlings and J. Neubueser of the Lehrstuhl D für Mathematik, Germany, for their tireless support. Moreover, we thank Prof. M. Artin of the Dept. of Mathematics, MIT, for his helpful insights and Prof. J. Humphreys of the University of Liverpool, UK, for comments. In addition we are indebted to P. Aspinwall, G.

Bertoldi, B. Fiol and N. Moeller for discussions as well as B. Pearre and M. Warden of the Dept. of Brain and Cognitive Sciences, MIT, for lending a hand at computer resources. We are also grateful to M. Warden for suggestions to the last draft of the manuscript. Finally we are particularly obliged to D. Berenstein for careful examination of the first versions of the paper and pointing out errors therein.

6. Appendix A: Some Explicit Computations for $M(G)$

6.1 Preliminary Definitions

We begin with a few rudimentary definitions [20]. Let H be a subgroup of G and let $g \in G$. For any cocycle $\alpha \in Z^2(G, \mathbb{C}^*)$ we define an induced action $g \cdot \alpha \in Z^2(gHg^{-1}, \mathbb{C}^*)$ thereon as $g \cdot \alpha(x,y) = \alpha(g^{-1}xg, g^{-1}yg), \;\; \forall \;\; x, y \in gHg^{-1}$. Now, it can be proved that the mapping

$$c_g : M(H) \to M(gHg^{-1}), \quad c_g(\alpha) := g \cdot \alpha$$

is a homomorphism, which we call **cocycle conjugation** by g.

On the other hand we have an obvious concept of restriction: for $S \subseteq L$ subgroups of G, we denote by $\mathrm{Res}_{L,S}$ the restriction map $M(L) \to M(S)$. Thereafter we define stability as:

DEFINITION 6.4 *Let H and K be arbitrary subgroups of G. An element $\alpha \in M(H)$ is said to be* **K-stable** *if*

$$\mathrm{Res}_{H, gHg^{-1} \cap H}(\alpha) = \mathrm{Res}_{gHg^{-1}, gHg^{-1} \cap H}(c_g(\alpha)) \;\; \forall \;\; g \in K.$$

The set of all K-stable elements of $M(H)$ will be denoted by $M(H)^K$ and it forms a subgroup of $M(H)$ known as the K-stable subgroup of $M(H)$.

When $K \subseteq N_G(H)$ all the above concepts[9] coalesce and we have the following important lemma:

LEMMA 6.1 ([20] p299) *If H and K are subgroups of G such that $K \subseteq N_G(H)$, then $M(H)^K$ is the K-stable subgroup of $M(H)$ with respect to the action of K on $M(H)$ induced by the action of K on H by conjugation. In other words,*

$$M(H)^K = \{\alpha \in M(H), \;\; \alpha(x,y) = c_g(\alpha)(x,y) \;\; \forall \;\; g \in K, \;\; \forall \;\; x, y \in H\}.$$

Finally let us present a useful class of subgroups:

DEFINITION 6.5 *A subgroup H of a group G is called a* **Hall subgroup** *of G if the order of H is coprime with its index in G, i.e.* $\gcd(|H|, |G/H|) = 1$.

[9] $N_G(H)$ is the normalizer of H in G, i.e., the set of all elements $g \in G$ such that $gHg^{-1} = H$. When H is a normal subgroup of G we obviously have $N_G(H) = G$.

For these subgroups we have:

THEOREM 6.6 ([20] p334) *If N is a normal Hall subgroup of G. Then*

$$M(G) \cong M(N)^{G/N} \times M(G/N).$$

The above theorem is really a corollary of a more general case of semi-direct products:

THEOREM 6.7 ([22] p33) *Let $G = N \rtimes T$ with $N \triangleleft G$, then*
(i) $M(G) \cong M(T) \times \tilde{M}(G)$;
(ii) The sequence $1 \to H^1(T, N^*) \to \tilde{M}(G) \stackrel{\text{Res}}{\to} M(N)^T \to H^2(T, N^*)$ *is exact, where* $\tilde{M}(G) := \ker \text{Res}_{G,N}$, $N^* := \text{Hom}(N, \mathbb{C}^*)$ *and* $H^{i=1,2}(T, N^*)$ *is the cohomology defined with respect to the conjugation action by* T *on* N^*.

Part (ii) of this theorem actually follows from the Lyndon-Hochschild-Serre spectral sequence into which we shall not delve.

One clarification is needed at hand. Let us define the first A-valued cohomology group for G, which we shall utilise later in our calculations. Here the 1-cocycles are the set of functions $Z^1(G, A) := \{f : G \to A | f(xy) = (x \cdot f(y))f(x) \quad \forall x, y \in G\}$, where A is being acted upon ($x \cdot A \to A$ for $x \in G$) by G as a $\mathbb{Z}G$-module. These are known as *crossed homomorphisms*. On the other hand, the 1-coboundaries are what is known as the principal crossed homomorphisms, $B^1(G, A) := \{f_{a \in A}(x) = (x \cdot a)a^{-1}\}$ from which we define $H^1(G, A) := Z^1(G, A)/B^1(G, A)$.

Alas, *caveat emptor*, we have defined in subsection 2.2, $H^2(G, A)$. There, the action of G on A (as in the case of the Schur Multiplier) is taken to be trivial, we must be careful, in the ensuing, to compute with respect to non-trivial actions such as conjugation. In our case the conjugation action of $t \in T$ on $\chi \in \text{Hom}(N, \mathbb{C}^*)$ is given by $\chi(tnt^{-1})$ for $n \in N$.

6.2 The Schur Multiplier for Δ_{3n^2}

6.2.1 Case I: $\gcd(n, 3) = 1$

Thus equipped, we can now use theorem 6.6 at our ease to compute the Schur multipliers the first case of the finite groups Δ_{3n^2}. Recall that $\mathbb{Z}_n \times \mathbb{Z}_n \triangleleft \Delta(3n^2)$ or explicitly

$$\Delta_{3n^2} \cong (\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes \mathbb{Z}_3.$$

Our crucial observation is that when $\gcd(n, 3) = 1$, $\mathbb{Z}_n \times \mathbb{Z}_n$ is in fact a normal Hall subgroup of Δ_{3n^2} with quotient group $\mathbb{Z}_3$. Whence Theorem 6.6 can be immediately applied to this case when n is coprime to 3:

$$M(\Delta_{3n^2}) = (M(\mathbb{Z}_n \times \mathbb{Z}_n))^{\mathbb{Z}_3} \times M(\mathbb{Z}_3) = (M(\mathbb{Z}_n \times \mathbb{Z}_n))^{\mathbb{Z}_3},$$

by recalling that the Schur Multiplier of all cyclic groups is trivial and that of $\mathbb{Z}_n \times \mathbb{Z}_n$ is $\mathbb{Z}_n$ [20]. But, $\mathbb{Z}_3 \subseteq N_{\Delta_{3n^2}}(\mathbb{Z}_n \times \mathbb{Z}_n) = \Delta_{3n^2}$, and hence by Lemma 6.1 it suffices to compute the $\mathbb{Z}_3$-stable subgroup of $\mathbb{Z}_n$ by cocycle conjugation.

Let the quotient group $\mathbb{Z}_3$ be $\langle z | z^3 = \mathbb{I}\rangle$ and similarly, if $x, y, x^n = y^n = \mathbb{I}$ are the generators of $\mathbb{Z}_n \times \mathbb{Z}_n$, then a generic element thereof becomes $x^a y^b, a, b = 0, \ldots, n-1$. The group conjugation by z on such an element gives

$$z^{-1} x^a y^b z = x^b y^{-a-b} \qquad z x^a y^b z^{-1} = x^{-a-b} y^a. \tag{6.1}$$

It is easy now to check that if α is a generator of the Schur multiplier $\mathbb{Z}_n$, we have an induced action

$$c_z(\alpha)(x^a y^b, x^{a'} y^{b'}) := \alpha(z^{-1} x^a y^b z, z^{-1} x^{a'} y^{b'} z) = \alpha(x^b y^{-(a+b)}, x^{b'} y^{-(a'+b')})$$

by Lemma 6.1.

However, we have a well-known result [11]:

PROPOSITION 6.2 *For the group $\mathbb{Z}_n \times \mathbb{Z}_n$, the explicit generator of the Schur Multiplier is given by*

$$\alpha(x^a y^b, x^{a'} y^{b'}) = \omega_n^{ab' - a'b}.$$

Consequently, $\alpha(x^b y^{-(a+b)}, x^{b'} y^{-(a'+b')}) = \alpha(x^a y^b, x^{a'} y^{b'})$ whereby making the c_z-action trivial and causing $(M(\mathbb{Z}_n \times \mathbb{Z}_n)^{\mathbb{Z}_3} \cong M(\mathbb{Z}_n \times \mathbb{Z}_n) = \mathbb{Z}_n$. From this we conclude part I of our result: $M(\Delta_{3n^2}) = \mathbb{Z}_n$ for n coprime to 3.

6.2.2 Case II: $\gcd(n, 3) \neq 1$

Here the situation is much more involved. Let us appeal to Part (ii) of Theorem 6.7. We let $N = \mathbb{Z}_n \times \mathbb{Z}_n$ and $T = \mathbb{Z}_3$ as above and define $U := \mathrm{Hom}(\mathbb{Z}_n \times \mathbb{Z}_n, \mathbb{C}^*)$); the exact sequence then takes the form

$$1 \to H^1(\mathbb{Z}_3, U) \to \tilde{M}(\Delta_{3n^2}) \to \mathbb{Z}_n \to H^2(\mathbb{Z}_3, U) \tag{6.2}$$

using the fact that the stable subgroup $M(\mathbb{Z}_n \times \mathbb{Z}_n)^{\mathbb{Z}_3} \cong \mathbb{Z}_n$ as shown above. Some explicit calculations are now called for.

As for U, it is of course isomorphic to $\mathbb{Z}_n \times \mathbb{Z}_n$ since for an Abelian group A, $\mathrm{Hom}(A, \mathbb{C}^*) \cong A$ ([22] p17). We label the elements thereof as $(p, q)(x^a y^b) := \omega_n^{ap+bq}$, taking $x^a y^b \in \mathbb{Z}_n \times \mathbb{Z}_n$ to $\mathbb{C}^*$.

We recall that the conjugation by $z \in \mathbb{Z}_3$ on $\mathbb{Z}_n \times \mathbb{Z}_n$ is (6.1). Therefore, by the remark at the end of the previous subsection, z acts on U as: $(z \cdot (p, q))(x^a y^b) := (p, q)(z(x^a y^b) z^{-1}) =$

$\omega_n^{a'p+b'q}$ with $a' = -a-b$ and $b' = a$ due[10] to (6.1), whence

$$z \cdot (p,q) = (q-p,-p), \qquad \text{for } (p,q) \in U. \tag{6.3}$$

Some explicit calculations are called for. First we compute $H^1(\mathbb{Z}_3, U)$. Z^1 is generically composed of functions such that $f(z) = (p,q)$ (and also $f(\mathbb{I}) = \mathbb{I}$ and $f(z^2) = (z \cdot f(z))f(z)$ by the crossed homomorphism condition, and is subsequently equal to $(q, p+q)$ by (6.3). Since no further conditions can be imposed, $Z^1 \cong \mathbb{Z}_n \times \mathbb{Z}_n$. Now B^1 consists of all functions of the form $(z\cdot(p,q))(p,q)^{-1} = (q-2p, -p-q)$, these are to be identified with the trivial map in Z^1. We can re-write these elements as $(p' := q-2p, -p'-3p) = (\omega_n^a\omega_n^{-b})^{p'}(\omega_n^b)^{-3p}$, and those in Z^1 we re-write as $(\omega_n^a\omega_n^{-b})^{p'}(\omega_n^b)^{q'}$ as we are free to do. Therefore if $\gcd(3,n) = 1$, then $H^1 := Z^1/B^1$ is actually trivial because in mod n, $3p$ also ranges the full $0, \cdots, n-1$, whereas if $\gcd(3,n) \neq 1$ then $H^1 := Z^1/B^1 \cong \mathbb{Z}_3$.

The computation for $H^2(\mathbb{Z}_3, U)$ is a little more involved, but the idea is the same. First we determine Z^2 as composed of $\alpha(z_1,z_2)$ constrained by the cocycle condition (with respect to conjugation which differs from (2.1) where the trivial action was taken)

$$\alpha(z_1,z_2)\alpha(z_1z_2,z_3) = (z_1 \cdot \alpha(z_2,z_3))\alpha(z_1,z_2z_3) \qquad z_1,z_2,z_3 \in \mathbb{Z}_3.$$

Again we only need to determine the following cases: $\alpha(z,z) := (p_1,q_1); \alpha(z^2,z^2) := (p_2,q_2);$ $\alpha(z^2,z) := (p_3,q_3); \alpha(z,z^2) := (p_4,q_4)$. The cocycle constraint gives $(p_1,q_1) = (q_4,-q_3);$ $(p_2,q_2) = (-q_3-q_4,-q_4); (p_3,q_3) = (-q_4,q_3); (p_4,q_4) = (p_4,q_4)$, giving $Z^2 \cong \mathbb{Z}_n \times \mathbb{Z}_n$. The coboundaries are given by $(\delta t)(z_1,z_2) = (z_1 \cdot t(z_2))t(z_1)t(z_1z_2)^{-1}$ (for any mapping $t : \mathbb{Z}_3 \to \mathbb{Z}_n \times \mathbb{Z}_n$ which we define to take values $t(z) = (r_1,s_1)$ and $t(z^2) = (r_2,s_2)$)), making $(\delta t)(z,z) = (s_1-r_2,-r_1+s_1-s_2); (\delta t)(z^2,z^2)(-s_2+r_2-r_1,r_2-s_1); (\delta t)(z^2,z) = (-s_1+r_2,r_1-s_1+s_2); (\delta t)(z,z^2) = (s_2-r_2+r_1,s_1-r_2)$. Now, the transformation $r_2 = s_1+q_4; r_1 = s_1-s_2-p_4+q_4$ makes this set of values for B^2 completely identical to those in Z^2, whence we conclude that $B^2 \cong \mathbb{Z}_n \times \mathbb{Z}_n$. In conclusion then $H^2 := Z^2/B^2 \cong \mathbb{I}$.

The exact sequence (6.2) then assumes the simple form of

$$1 \to \left\{ \begin{array}{c} \mathbb{Z}_3,\ \gcd(n,3) \neq 1 \\ \mathbb{I},\ \gcd(n,3) = 1 \end{array} \right\} \to \tilde{M}(G) \to \mathbb{Z}_n \to 1,$$

which means that if n does not divide 3, $\tilde{M}(G) \cong \mathbb{Z}_n$, and otherwise $\tilde{M}(G)/\mathbb{Z}_3 \cong \mathbb{Z}_n$. Of course, in conjunction with Part (i) of Theorem 6.7, we immediately see that the first case makes Part I of our discussion (when $\gcd(n,3) = 1$) a special case of our present situation.

On the other hand, for the remaining case of $\gcd(n,3) \neq 1$, we have $M(\Delta_{3n^2})/\mathbb{Z}_3 \cong \mathbb{Z}_n$, which means that $M(\Delta_{3n^2})$, being an Abelian group, can only be $\mathbb{Z}_{3n}$ or $\mathbb{Z}_n \times \mathbb{Z}_3$. The

[10] Note that we must be careful to let the order of conjugation be the opposite of that in the cocycle conjugation.

exponent of the former is $3n$, while the later (since 3 divides n), is n, but by Theorem 3.4, the exponent squared must divide the order, which is $3n^2$, whereby forcing the second choice.

Therefore in conclusion we have our *theorema egregium*:

$$M(\Delta_{3n^2}) = \begin{cases} \mathbb{Z}_n \times \mathbb{Z}_3, & \gcd(n,3) \neq 1 \\ \mathbb{Z}_n, & \gcd(n,3) = 1 \end{cases}$$

as reported in Table (3.2).

6.3 The Schur Multiplier for Δ_{6n^2}

Recalling that n is even, we have $\Delta_{6n^2} \cong (\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes S_3$ with $\mathbb{Z}_n \times \mathbb{Z}_n$ normal and thus we are once more aided by Theorem 6.7.

We let $N := \mathbb{Z}_n \times \mathbb{Z}_n$ and $T := S_3$ and the exact sequence assumes the form

$$1 \to H^1(S_3, U) \to \tilde{M}(\Delta_{6n^2}) \to (\mathbb{Z}_n)^{S_3} \to H^2(S_3, U)$$

where $U := \mathrm{Hom}(\mathbb{Z}_n \times \mathbb{Z}_n, \mathbb{C}^*)$ as defined in the previous subsection.

By calculations entirely analogous to the case for Δ_{3n^2}, we have $(\mathbb{Z}_n)^{S_3} \cong \mathbb{Z}_2$. This is straight-forward to show. Let $S_3 := \langle z, w | z^3 = w^2 = \mathbb{I}, zw = wz^2 \rangle$. We see that it contains $\mathbb{Z}^3 = \langle z | z^3 = \mathbb{I} \rangle$ as a subgroup, which we have treated in the previous section. In addition to (6.1), we have

$$w^{-1} x^a y^b w = x^{-1-b} y^b = w x^a y^b w^{-1}.$$

Using the form of the cocycle in Proposition 6.2, we see that $c_w(\alpha) = \alpha^{-1}$. Remembering that $c_z(\alpha) = \alpha$ from before, we see that the S_3-stable part of consists of α^m with $m = 0$ and $n/2$ (recall that in our case of $\Delta(6n^2)$, n is even), giving us a $\mathbb{Z}_2$.

Moreover we have $H^1(S_3, U) \cong \mathbb{I}$. This is again easy to show. In analogy to (6.3), we have

$$w \cdot (p, q) = (-q, q - p), \qquad \text{for } (p, q) \in U,$$

using which we find that Z^1 consists of $f : S_3 \to U$ given by $f(z) = (l_1, 3k_2 - l_1)$ and $f(w) = (2k_2, k_2)$. In addition B^1 consists of $f(z) = (k - 2l, -l - k)$ and $f(w) = (-2l, -l)$. Whence we see instantly that H^1 is trivial.

Now in fact $H^2(S_3, U) \cong \mathbb{I}$ as well (the involved details of these computations are too pathological to be even included in an appendix and we have resisted the urge to write an appendix for the appendix).

The exact sequence then forces immediately that $\tilde{M}(\Delta_{6n^2}) \cong \mathbb{Z}_2$. Moreover, since $M(S_3) \cong \mathbb{I}$ (q.v. e.g. [20]), by Part (i) of Theorem 6.7, we conclude that

$$M(\Delta_{6n^2}) \cong \mathbb{Z}_2$$

as reported in Table (3.2).

7. Appendix B: Intransitive subgroups of $SU(3)$

The computation of the Schur Multipliers for the non-Abelian intransitive subgroups of $SU(3)$ involves some subtleties related to the precise definition and construction of the groups.

Let us consider the case of combining the generators of $\mathbb{Z}_n$ with these of $\widehat{D_{2m}}$ to construct the intransitive subgroup $<\mathbb{Z}_n, \widehat{D_{2m}}>$. We can take the generators of $\widehat{D_{2m}}$ to be

$$\alpha = \begin{pmatrix} \omega_{2m} & 0 & 0 \\ 0 & \omega_{2m}^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and that of $\mathbb{Z}_n$ to be

$$\gamma = \begin{pmatrix} \omega_n & 0 & 0 \\ 0 & \omega_n & 0 \\ 0 & 0 & \omega_n^{-2} \end{pmatrix}.$$

The group $<\mathbb{Z}_n, \widehat{D_{2m}}>$ is not in general the direct product of $\mathbb{Z}_n$ and $\widehat{D_{2m}}$. More specifically, when n is odd $<\mathbb{Z}_n, \widehat{D_{2m}}> = \mathbb{Z}_n \times \widehat{D_{2m}}$. For n even however, we notice that $\alpha^m = \beta^2 = \gamma^{n/2}$. Accordingly, we conclude that $<\mathbb{Z}_n, \widehat{D_{2m}}> = (\mathbb{Z}_n \times \widehat{D_{2m}})/\mathbb{Z}_2$ for n even where the central $\mathbb{Z}_2$ is generated by $\gamma^{n/2}$. Actually the conditions are more refined: when $n = 2(2k+1)$ we have $\mathbb{Z}_n = \mathbb{Z}_2 \times \mathbb{Z}_{2k+1}$ and so $(\mathbb{Z}_2 \times \widehat{D_{2m}})/\mathbb{Z}_2 = \mathbb{Z}_{2k+1} \times \widehat{D_{2m}}$. Thus the only non-trivial case is when $n = 4k$.

This subtlety in the group structure holds for all the cases where $\mathbb{Z}_n$ is combined with binary groups $\widehat{G}$. When $n \bmod 4 \neq 0$, $<\mathbb{Z}_n, \widehat{G}>$ is the direct product of $\widehat{G}$ with either $\mathbb{Z}_n$ or $\mathbb{Z}_{n/2}$. For $n \bmod 4 = 0$ it is the quotient group $(\mathbb{Z}_n \times \widehat{G})/\mathbb{Z}_2$. In summary

$$<\mathbb{Z}_n, \widehat{G}> = \begin{cases} \mathbb{Z}_n \times \widehat{G} & n \bmod 2 = 1 \\ \mathbb{Z}_{n/2} \times \widehat{G} & n \bmod 4 = 2 \\ (\mathbb{Z}_n \times \widehat{G})/\mathbb{Z}_2 & n \bmod 4 = 0 \end{cases}.$$

The case of $\mathbb{Z}_n$ combined with the ordinary dihedral group D_{2m} is a bit different however. The matrix forms of the generators are

$$\alpha = \begin{pmatrix} \omega_m & 0 & 0 \\ 0 & \omega_m^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad \gamma = \begin{pmatrix} \omega_n & 0 & 0 \\ 0 & \omega_n & 0 \\ 0 & 0 & \omega_n^{-2} \end{pmatrix}$$

where α and β generate D_{2m} and γ generates $\mathbb{Z}_n$.

From these we notice that when both n and m are even, $\alpha^{m/2} = \gamma^{n/2}$ and $< \mathbb{Z}_n, D_{2m} >$ is not a direct product. After inspection, we find that

$$< \mathbb{Z}_n, D_{2m} >= \begin{cases} \mathbb{Z}_n \times D_{2m} & m \bmod 2 = 1 \\ \mathbb{Z}_n \times D_{2m} & m \bmod 2 = 0, n \bmod 2 = 1 \\ \mathbb{Z}_{n/2} \times D_{2m} & m \bmod 2 = 0, n \bmod 4 = 2 \\ (\mathbb{Z}_n \times D_{2m})/\mathbb{Z}_2 & m \bmod 2 = 0, n \bmod 4 = 0 \end{cases}.$$

The Schur Multipliers of the direct product cases are immediately computable by consulting Theorem 3.5. For example, $M(\mathbb{Z}_n \times \widehat{D_{2m}}) \cong M(\mathbb{Z}_n) \times M(\widehat{D_{2m}}) \times (\mathbb{Z}_n \otimes \widehat{D_{2m}})$ by Theorem 3.5, the last term of which in turn equates to $\mathrm{Hom}(\mathbb{Z}_n, \widehat{D_{2m}}/\widehat{D_{2m}}')$. This is $\mathrm{Hom}(\mathbb{Z}_n, \mathbb{Z}_2 \times \mathbb{Z}_2) \cong \mathbb{Z}_{\gcd(n,2)} \times \mathbb{Z}_{\gcd(n,2)}$ for m even and $\mathrm{Hom}(\mathbb{Z}_n, \mathbb{Z}_4) \cong \mathbb{Z}_{\gcd(n,4)}$ for m odd. By similar token, we have that $M(\mathbb{Z}_n \times D_{2m})$ for even m is $\mathbb{Z}_2 \times \mathrm{Hom}(\mathbb{Z}_n, \mathbb{Z}_2 \times \mathbb{Z}_2) \cong \mathbb{Z}_2 \times \mathbb{Z}_{\gcd(n,2)} \times \mathbb{Z}_{\gcd(n,2)}$ and $\mathrm{Hom}(\mathbb{Z}_n, \mathbb{Z}_2) \cong \mathbb{Z}_{\gcd(n,2)}$ for odd m. Likewise $M(\mathbb{Z}_n \times \widehat{E_{6,7,8}}) = \mathrm{Hom}(\mathbb{Z}_n, \mathbb{Z}_{3,2,1})$.

References

[1] A. Connes, M. Douglas and A. Schwarz, "Noncommutative Geometry and Matrix Theory: Compactification on Tori," hep-th/9711162;
M. Douglas and C. Hull, "D-branes and the Noncommutative Torus," hep-th/9711165;
N. Seiberg and E. Witten, "String Theory and Noncommutative Geometry," hep-th/9908142.

[2] C. Vafa, "Modular Invariance and Discrete Torsion on Orbifolds," Nuc. Phys. B273 (1986).

[3] C. Vafa and E. Witten, "On Orbifolds with Discrete Torsion," hep-th/9409188.

[4] M. Douglas, "D-branes and Discrete Torsion," hep-th/9807235.

[5] M. Douglas and B. Fiol, "D-branes and Discrete Torsion II," hep-th/9903031.

[6] M. Douglas and G. Moore, "D-branes, Quivers, and ALE Instantons," hep-th/9603167.

[7] A. Lawrence, N. Nekrasov and C. Vafa, "On Conformal Field Theories in Four Dimensions," hep-th/9803015.

[8] D. Berenstein and R. Leigh, "Discrete Torsion, AdS/CFT and Duality," hep-th/0001055.

[9] D. Berenstein, V. Jejjala and R. Leigh, "Marginal and Relevant Deformations of N=4 Field Theories and Non-Commutative Moduli Spaces of Vacua," hep-th/0005087;
–, "Non-Commutative Moduli Spaces and T-duality," hep-th/0006168;
D. Berenstein and R. Leigh, "Non-Commutative Calabi-Yau Manifolds," hep-th/0009209.

[10] S. Mukhopadhyay and K. Ray, "D-branes on Fourfolds with Discrete Torsion," hep-th/9909107.

[11] M. Klein and R. Rabadan, "Orientifolds with discrete torsion," hep-th/0002103;
–, "$Z_N \times Z_M$ orientifolds with and without discrete torsion," hep-th/0008173.

[12] K. Dasgupta, S. Hyun, K. Oh and R. Tatar, "Conifolds with Discrete Torsion and Noncommutativity," hep-th/0008091.

[13] M. Gaberdiel, "Discrete torsion, orbifolds and D-branes," hep-th/0008230.

[14] E. Sharpe, 'Discrete Torsion and Gerbes I," hep-th/9909108;
–, "Discrete Torsion and Gerbes II," hep-th/9909120;
–, "Discrete Torsion," hep-th/0008154;
–, 'Analogues of Discrete Torsion for the M-Theory Three-Form," hep-th/0008170;
–, "Discrete Torsion in Perturbative Heterotic String Theory," hep-th/0008184;
–, "Recent Developments in Discrete Torsion," hep-th/0008191.

[15] Y.-H. He and J. S. Song, "Of McKay Correspondence, Non-linear Sigma-model and Conformal Field Theory," hep-th/9903056.

[16] J. Gomis, "D-branes on Orbifolds with Discrete Torsion And Topological Obstruction," hep-th/0001200.

[17] P. Aspinwall and M. R. Plesser, "D-branes, Discrete Torsion and the McKay Correspondence," hep-th/0009042.

[18] P. Aspinwall, "A Note on the Equivalence of Vafa's and Douglas's Picture of Discrete Torsion," hep-th/0009045.

[19] A. Kapustin, "D-branes in a topologically nontrivial B-field," hep-th/9909089.

[20] G. Karpilovsky, "Group Representations" Vol. II, Elsevier Science Pub. 1993.

[21] G. Karpilovsky, "Projective Representations of Finite Groups," Pure and Applied Math. 1985.

[22] G. Karpilovsky, "The Schur Multiplier," London Math. Soc. Monographs, New Series 2, Oxford 1987.

[23] B. Feng, A. Hanany, and Y.-H. He, "Z-D Brane Box Models and Non-Chiral Dihedral Quivers," hep-th/9909125.

[24] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.2, Aachen, St Andrews, 1999, http://www.math.rwth-aachen.de/~GAP/WWW/gap.html.

[25] D. Holt, "The Calculation of the Schur Multiplier of a Permutation Group," Proc. Lond. Math. Soc. Meeting on Computational Group Theory (Durham 1982) Ed. M. Atkinson.

[26] W. M. Fairbairn, T. Fulton and W. Klink, "Finite and Disconnected Subgroups of SU_3 and their Applications to The Elementary Particle Spectrum," J. Math. Physics, vol 5, Number 8, 1964, pp1038 - 1051.

[27] A. Hanany and Y.-H. He, "Non-Abelian Finite Gauge Theories," hep-th/9811183.

[28] T. Muto, "D-branes on Three-dimensional Nonabelian Orbifolds," hep-th/9811258.

[29] A. Hanany and Y.-H. He, "A Monograph on the Classification of the Discrete Subgroups of SU(4)," hep-th/9905212.

[30] Clifford V. Johnson, Robert C. Myers, "Aspects of Type IIB Theory on ALE Spaces," hep-th/9610140.

[31] S. Dulat, K. Wendland, "Crystallographic Orbifolds: Towards a Classification of Unitary Conformal Field Theories with Central Charge c = 2," hep-th/0002227.

[32] R. Dijkgraaf, "Discrete Torsion and Symmetric Products," hep-th/9912101;
P. Bantay, "Symmetric Products, Permutation Orbifolds and Discrete Torsion", hep-th/0004025.

[33] D. Berenstein, Private communications.